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Analyticity of QCD observables beyond leading-order perturbation theory¹

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Abstract

A theoretical framework is presented to treat hadronic observables within analytic perturbative QCD beyond the leading order of the coupling and for more than one single large momentum scale. The approach generalizes and extends the pioneering work of Shirkov and Solovtsov on an analytic strong running coupling. Some applications to hadronic observables at the partonic level are also discussed.

1 Homage to Igor Solovtsov

I (NGS) met Igor for the first time many years ago in the beginning of the nineties during a physics conference in Dubna. It was a chance encounter

¹Invited contribution in memory of I.L. Solovtsov at Seminar in Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia, Jan. 17, 2008.

during the conference dinner, when it happened to get a seat next to him. Igor introduced himself and I took the opportunity to discuss with him about variational perturbation theory and its convergence properties, a subject on what he was working at that time, known to me from his publications. We soon found out that we had a lot of common interests in physics and we kept discussing for hours, while emptying a bottle of vodka. During this first encounter, I could, of course, not imagine that years later I would engross myself so strongly in Analytic Perturbation Theory, a subject pioneered by Igor and Dmitry V. Shirkov, as it actually happened. This activity will be surveyed in this invited contribution to his memory. Since then, I met Igor on several occasions, in Dubna and abroad, and I still have a strong recollection of our discussions. Igor indelibly forged his name in the annals of physics by his scientific achievements—no doubt. But those of us, who had the privilege to know him in person, will sadly remember and miss his intellectual creativity along with his kindness.

2 Introduction

Traditionally, perturbation theory in QCD suffers from an artificial singularity at momenta close to Λ_{QCD} , called (at the one-loop level) the Landau pole. Since the early days of QCD, theorists have tried different remedies to avert this problem, like infrared (IR) cutoffs, “freezing”, etc. But approximately ten years ago, Shirkov and Solovtsov have devised an approach that avoids this problem by appealing only to a few basic principles of Quantum Field Theory—chief among them, Causality and Renormalizability—and avoiding the introduction of extraneous infrared (IR) regulators. Since this pioneering work appeared in the year 1996 [1, 2], the analytic approach to QCD perturbation theory has evolved and considerably progressed, shifting the cutting edge significantly (see [3–5] for reviews and further references), and finally culminating into the so-called Analytic Perturbation Theory (APT). Meanwhile this analytic approach has been extended beyond the one-loop level [6, 7] and important techniques for numerical calculations have been developed [8–13]. Also applications to the ultra-low momentum region have been carried out, e.g., [14], and alternative formulations of the strong coupling below the Landau pole have been proposed aiming to incorporate nonperturbative input [15, 16].

The simple analytization concept of the strong running coupling has been generalized to the level of hadronic amplitudes [17, 18] and new techniques have been developed to deal with more than one large hard scale in the process [19, 20], including also Sudakov resummation in exclusive processes. In

the later course of these investigations, it was realized that logarithms of the aforementioned second large scale—which can be the factorization or the evolution scale—correspond to non-integer (fractional) powers of the coupling, giving rise to Fractional Analytic Perturbation Theory (or FAPT for short) [21, 22]. At the heart of this development was the Karanikas-Stefanis (KS) [17] analytization principle which demands that all terms in a QCD amplitude that can affect the discontinuity across the cut along the negative real axis $-\infty < Q^2 < 0$, and hence contribute to the spectral density, have to be included into the analytization procedure, i.e., the dispersion relation. The KS procedure encompasses the Shirkov-Solovtsov analytization concept of integer powers of the strong coupling and paves the way to the analytization of any real power both in the Euclidean [21] as well as and in the Minkowski region [23]. The crucial advantages of this scheme are:

- A diminished sensitivity on the factorization scale of typical QCD hard processes, like the factorized part of the pion’s electromagnetic form factor—verified in [22] to the next-to-leading order (NLO).
- A quasi renormalization-scale setting and scheme independence of the same observable at the NLO level [24] (see for an abridged version [25]).
- The inclusion of evolution effects due to the running of the strong coupling beyond the one-loop order [23].
- A faster convergence of the perturbative expansion in terms of analytic images of real powers of the strong coupling [21].
- The resummation to all orders of the π^2 terms, induced by analytic continuation into the timelike domain. This has been exemplarily verified in [23] for the scalar Higgs decay into a $b\bar{b}$ pair at the four-loop level.

The most important consequence of the analytic approach is that it ties the abstract mathematical requirements of causality and renormalizability to something tangible, like the calculation of the Bjorken [26] and the Gross-Llewellyn Smith sum rule [27], or the inclusive decay of a τ -lepton into hadrons [28–30], the pion’s electromagnetic form factor [19, 20, 22, 24], the Higgs boson decay into a $b\bar{b}$ pair [23], and many other processes. In the present exposition we will present the bedrock of this approach, focusing our attention to selected applications and results beyond the leading order (LO) of perturbative QCD—conventional and analytic.

3 Basic Structure of FAPT

Let us briefly review FAPT, specifying our notation, explaining its main principles, and describing its methodology.

The running strong coupling in QCD

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0} a_s(L) \quad \text{with} \quad L \equiv \ln \frac{Q^2}{\Lambda^2}, \quad (1)$$

where Λ denotes the characteristic scale of QCD, Λ_{QCD} , satisfies the renormalization-group (RG) equation

$$\frac{d}{dL} \left(\frac{\alpha_s}{4\pi} \right) = \beta \left(\frac{\alpha_s}{4\pi} \right) = -b_0 \left(\frac{\alpha_s}{4\pi} \right)^2 - b_1 \left(\frac{\alpha_s}{4\pi} \right)^3 - b_2 \left(\frac{\alpha_s}{4\pi} \right)^4 - \dots \quad (2)$$

with known β -function coefficients up to the displayed order. [Their explicit expressions can be found, for instance, in [23]]. At the one-loop order, $a_s(L)$ develops a Landau pole, while the two-loop solution of Eq. (2) has a square-root singularity, with more complicated singularities for still higher orders. Shirkov and Solovtsov [2] have shown that the ghost-singularity problem can be solved only on account of renormalizability—in terms of the RG equation—and causality—expressed in the form of a dispersion relation. Then, one obtains in Euclidean space analytic images of the coupling at loop order l

$$\mathcal{A}_m^{(l)}(Q^2) \equiv \left[a_{(l)}^m(Q^2) \right]_{\text{an}} \quad (3)$$

following from the Källén-Lehmann spectral representation

$$\left[f(Q^2) \right]_{\text{an}} = \frac{1}{\pi} \int_0^\infty \frac{\mathbf{Im} \left[f(-\sigma) \right]}{\sigma + Q^2 - i\epsilon} d\sigma \quad (4)$$

with the spectral density at one loop given by (see [21, 23] for higher loops)

$$\rho_\nu(\sigma) = \frac{1}{\pi} \mathbf{Im} \left[a^\nu(-\sigma) \right] = \frac{1}{\pi} \frac{\sin(\nu\phi)}{[\pi^2 + L^2(\sigma)]^{\nu/2}}. \quad (5)$$

As a result, one then finds in the Euclidean space at one loop

$$\mathcal{A}_1^{(1)}(Q^2) = \frac{1}{L} - \frac{1}{e^L - 1}, \quad (6)$$

while its counterpart in Minkowski space reads

$$\mathfrak{A}_1^{(1)}(s) = \frac{1}{\pi} \arccos \left(\frac{L_s}{\sqrt{L_s^2 + \pi^2}} \right) \quad (7)$$

with $L_s = \ln(s/\Lambda^2)$. This procedure is sufficient in considering QCD processes with one large scale, but fails for non-integer powers of a_s and is unable to accommodate terms, like

- $[a_s(L)]^{\gamma_0/2\beta_0} \longleftrightarrow$ RG at one loop
- $[a_s(L)]^n \ln[a_s(L)] \longleftrightarrow$ RG at two loops
- $[a_s(L)]^n L^m \longleftrightarrow$ Factorization
- $\exp[-a_s(L)F(x)] \longleftrightarrow$ Sudakov resummation (symbolically)

which typically appear in perturbative calculations beyond the leading order due to the reasons already explained. Such terms do not modify the ghost singularities but they do contribute to the spectral density and, hence, their analytic images are inevitably required for the dispersion relation. In actual fact, precisely such terms are tantamount to *fractional* (real) powers of the strong coupling [21, 22].

The core feature of FAPT is, as mentioned in the Introduction, the KS analytization principle [17]. The use of this principle allows the inclusion into the dispersion relations of logarithmic terms, of the sort $\ln(Q^2/\mu_F^2)$, or products of such logarithms with powers of the running coupling. To appreciate its meaning and usefulness, we present and compare different analytization concepts in Fig. 1.

In this figure, the linear operations \mathbf{A}_E and \mathbf{A}_M define, respectively, the analytic running couplings in the Euclidean (spacelike)

$$\mathbf{A}_E \left[a_{(l)}^n \right] = \mathcal{A}_n^{(l)} \quad \text{with} \quad \mathcal{A}_n^{(l)}(Q^2) \equiv \int_0^\infty \frac{\rho_n^{(l)}(\sigma)}{\sigma + Q^2} d\sigma \quad (8)$$

and the Minkowski (timelike) region

$$\mathbf{A}_M \left[a_{(l)}^n \right] = \mathfrak{A}_n^{(l)} \quad \text{with} \quad \mathfrak{A}_n^{(l)}(s) \equiv \int_s^\infty \frac{\rho_n^{(l)}(\sigma)}{\sigma} d\sigma. \quad (9)$$

The above analytization operations can be represented by the following two integral transformations from the timelike region to the spacelike region (see, e.g., [4]):

$$\hat{D}[\mathfrak{A}_n^{(l)}] = \mathcal{A}_n^{(l)} \quad \text{with} \quad \mathcal{A}_n^{(l)}(Q^2) \equiv Q^2 \int_0^\infty \frac{\mathfrak{A}_n^{(l)}(\sigma)}{(\sigma + Q^2)^2} d\sigma \quad (10)$$

and for the inverse transformation

$$\hat{R}[\mathcal{A}_n^{(l)}] = \mathfrak{A}_n^{(l)} \quad \text{with} \quad \mathfrak{A}_n^{(l)}(s) \equiv \frac{1}{2\pi i} \int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{\mathcal{A}_n^{(l)}(\sigma)}{\sigma} d\sigma. \quad (11)$$

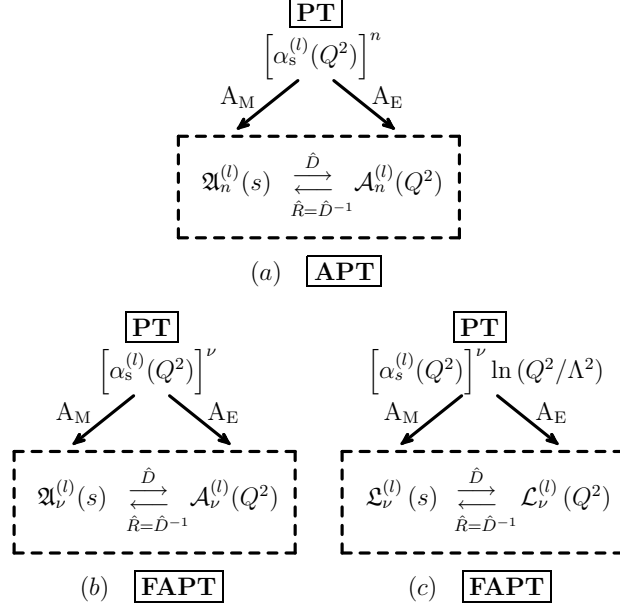


Fig. 1: Illustration of different analytization concepts. (a) APT (b) FAPT, only strong-coupling powers, (c) FAPT, products of strong-coupling powers and logarithms. In APT, the index n is restricted to integer values only; in FAPT ν can assume any real value. Further explanations are given in [23].

These two integral transformations are connected to each other by the relation

$$\hat{D}\hat{R} = \hat{R}\hat{D} = 1, \quad (12)$$

valid for the whole set of analytic images of the powers of the coupling in the Euclidean as well as in the Minkowski space, $\{\mathcal{A}_n, \mathfrak{A}_n\}$, respectively, and at any desired loop order of the perturbative expansion.

In the spacelike region, the analytic images of the coupling can be expressed in terms of the reduced transcendental Lerch function $F(z, \nu)$ to read [21] ($L \equiv \ln(Q^2/\Lambda^2)$)²

$$\mathcal{A}_\nu(L) = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1 - \nu)}{\Gamma(\nu)}, \quad (13)$$

where the first term corresponds to the conventional term of perturbative QCD and the second one is entailed by the pole remover (cf. $1/(e^L - 1)$) at one

²Everywhere in this presentation Greek labels denote non-integer (real) powers or indices.

loop). This function is an entire function in the index ν and has the properties $\mathcal{A}_0(L) = 1$, $\mathcal{A}_{-m}(L) = L^m$ for $m \in \mathbb{N}$, and $\mathcal{A}_m(\pm\infty) = 0$ for $m \geq 2$, $m \in \mathbb{N}$, while for $|L| < 2\pi$, it reads $\mathcal{A}_\nu(L) = -[1/\Gamma(\nu)] \sum_{r=0}^{\infty} \zeta(1-\nu-r) [(-L)^r/r!]$. In the timelike region, these images are completely determined by elementary functions [21] ($L_s \equiv \ln(s/\Lambda^2)$):

$$\mathfrak{A}_\nu(L_s) = \frac{\sin \left[(\nu-1) \arccos \left(L_s / \sqrt{\pi^2 + L_s^2} \right) \right]}{\pi(\nu-1) (\pi^2 + L_s^2)^{(\nu-1)/2}} \quad (14)$$

from which, for example, we get $\mathfrak{A}_0(L_s) = 1$ and $\mathfrak{A}_{-1}(L_s) = L_s$. The salient characteristics of FAPT in comparison with APT and the standard perturbative expansion in QCD are compiled in Table 1, while for further reading and graphic illustrations we refer the reader to [21–23].

Table 1: FAPT versus APT and standard QCD perturbation theory (SPT)

Theory	SPT	APT	FAPT
Space	$\{a^\nu\}_{\nu \in \mathbb{R}}$	$\{\mathcal{A}_m\}_{m \in \mathbb{N}}$	$\{\mathcal{A}_\nu\}_{\nu \in \mathbb{R}}$
Series expansion	$\sum_m f_m a^m(L)$	$\sum_m f_m \mathcal{A}_m(L)$	$\sum_m f_m \mathcal{A}_m(L)$
Inverse powers	$[a(L)]^{-m}$	—	$\mathcal{A}_{-m}(L) = L^m$
Multiplication	$a^\mu a^\nu = a^{\mu+\nu}$	—	—
Index derivative	$a^\nu \ln^k a$	—	$\frac{d^k \mathcal{A}_\nu}{d\nu^k} = [a^\nu \ln^k(a)]_{\text{an}}$

4 FAPT Applications at the NLO and Beyond

In this section, we concentrate on applications of the presented FAPT formalism to two QCD processes beyond the leading order of perturbative QCD.

The first process to be considered is the factorizable part of the pion’s electromagnetic form factor at NLO accuracy in Euclidean space. This process has been widely discussed in the literature using various techniques—see e.g., [24] for a recent comprehensive analysis and comparison with the available experimental data. At leading twist, one has the convolution $(A(z) \otimes zB(z) \equiv \int_0^1 dz A(z)B(z))$

$$F_\pi^{\text{Fact}}(Q^2) = \varphi_\pi(x, \mu_F^2) \otimes T_H^{\text{NLO}}(x, y, Q^2; \mu_F^2, \mu_R^2) \otimes \varphi_\pi(y, \mu_F^2), \quad (15)$$

where the twist-two pion distribution amplitude (DA) (using $\bar{x} \equiv 1 - x$)

$$\varphi_\pi(x, \mu^2) = 6x\bar{x} \left[1 + a_2(\mu^2) C_2^{3/2}(2x-1) + a_4(\mu^2) C_4^{3/2}(2x-1) + \dots \right] \quad (16)$$

contains all non-perturbative information on the pion quark structure in terms of the Gegenbauer coefficients a_n , determined at some typical hadronic scale $\mu^2 \approx 1 \text{ GeV}^2$ [31–33]. Note that the quantity $F_\pi^{\text{Fact}}(Q^2)$ depends, beyond the LO, on two scales: the factorization scale μ_F and the renormalization scale μ_R .

To appreciate the differences among the various analytization schemes, consider the scaled hard-scattering amplitude entering Eq. (15) in the so-called “Naive Analytization” (Naive-An) scheme [19, 20] in comparison with the “Maximal Analytization” (MA) scheme [24] (with the renormalization scale set equal to $\mu_R^2 = \lambda_R Q^2$, λ_R being a numerical parameter):

Naive Analytization

$$\begin{aligned} [Q^2 T_H(x, y, Q^2; \mu_F^2, \lambda_R Q^2)]_{\text{Naive-An}} &= \mathcal{A}_1^{(2)}(\lambda_R Q^2) t_H^{(0)}(x, y) \\ &+ \frac{[\mathcal{A}_1^{(2)}(\lambda_R Q^2)]^2}{4\pi} t_H^{(1)}\left(x, y; \lambda_R, \frac{\mu_F^2}{Q^2}\right) \end{aligned} \quad (17)$$

Maximal Analytization

$$\begin{aligned} [Q^2 T_H(x, y, Q^2; \mu_F^2, \lambda_R Q^2)]_{\text{Max-An}} &= \mathcal{A}_1^{(2)}(\lambda_R Q^2) t_H^{(0)}(x, y) \\ &+ \frac{\mathcal{A}_2^{(2)}(\lambda_R Q^2)}{4\pi} t_H^{(1)}\left(x, y; \lambda_R, \frac{\mu_F^2}{Q^2}\right). \end{aligned} \quad (18)$$

In these equations, $t_H^{(0)}(x, y)$ and $t_H^{(1)}\left(x, y; \lambda_R, \frac{\mu_F^2}{Q^2}\right)$ stand, respectively, for the LO and NLO hard-scattering amplitudes, computed in [34].

The Naive Analytization just replaces the strong coupling and its powers by their corresponding analytic images. This procedure is, strictly speaking, incorrect [20] because $[\mathcal{A}_1(L)]^n \neq [a_s^n(L)]_{\text{An}}$, owing to their distinct spectral representations. This scheme, whatever its theoretical shortcomings, works phenomenologically rather well [19, 20]. Its direct improvement in [24] adopts instead the Maximal Analytization, which associates to the powers of the running coupling their own dispersive images, trading this way the usual power series expansion for a non-power functional expansion, i.e., $[a_s^n(L)]_{\text{Max-An}} = \mathcal{A}_n(L)$. The difference between the two analytization schemes becomes apparent by comparing in the equations above the NLO terms. This theoretical improvement entails a phenomenological improvement as well. From Fig. 2 we see that the crucial advantage of the FAPT

analysis is that the dependence of the prediction for $F_\pi^{\text{Fact}}(Q^2)$ on the perturbative scheme and scale setting is diminished already at NLO. Next we will show that applying the KS analytization procedure, the result will become insensitive also to the variation of the factorization scale.

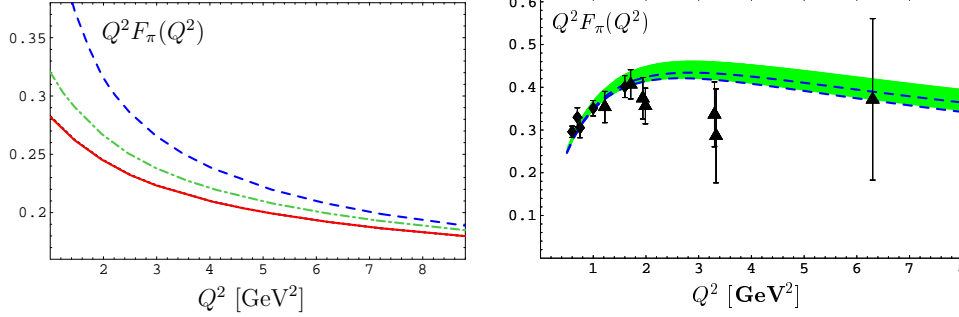


Fig. 2: (Left) Results for $Q^2 F_\pi^{\text{Fact}}(Q^2)$ vs. Q^2 with $\mu_R^2 = Q^2$, $\mu_F^2 = 5.76 \text{ GeV}^2$ in SPT of QCD (dashed line), using Naive Analytization (dash-dotted line), and with Maximal Analytization (solid line). (Right) The same quantity (in Max-An) in comparison with experimental data (see [24]). The broken lines denote the region accessible to the asymptotic pion DA, while the shaded strip marks the region of predictions derived with the pion DAs from nonlocal QCD sum rules [31] (cf. Eq. (16)).

Consistent with this requirement the analytization of the logarithmic term $\ln(Q^2/\mu_F^2) = \ln(\lambda_R Q^2/\Lambda^2) - \ln(\lambda_R \mu_F^2/\Lambda^2)$ has to be performed as well, so that after some manipulations, explained in [22], we obtain

$$\begin{aligned} [Q^2 T_H(x, y, Q^2; \mu_F^2, \lambda_R Q^2)]_{\text{KS}}^{\text{An}} &= \mathcal{A}_1^{(2)}(\lambda_R Q^2) t_H^{(0)}(x, y) \\ &+ \frac{\mathcal{A}_2^{(2)}(\lambda_R Q^2)}{4\pi} t_H^{(1)}\left(x, y; \lambda_R, \frac{\mu_F^2}{Q^2}\right) \\ &+ \frac{\Delta_2^{(2)}(\lambda_R Q^2)}{4\pi} \left[C_F t_H^{(0)}(x, y) (6 + 2 \ln(\bar{x}\bar{y})) \right], \end{aligned} \quad (19)$$

with the deviation from the second line in Eq. (18) being encoded in the term

$$\Delta_2^{(2)}(Q^2) \equiv \mathcal{L}_2^{(2)}(Q^2) - \mathcal{A}_2^{(2)}(Q^2) \ln[Q^2/\Lambda^2] \quad (20)$$

where

$$\mathcal{L}_2^{(2)}(Q^2) \equiv \left[\left(\alpha_s^{(2)}(Q^2) \right)^2 \ln\left(\frac{Q^2}{\Lambda^2}\right) \right]_{\text{KS}}^{\text{An}} = \frac{4\pi}{b_0} \left[\frac{\left(\alpha_s^{(2)}(Q^2) \right)^2}{\alpha_s^{(1)}(Q^2)} \right]_{\text{KS}}^{\text{An}}. \quad (21)$$

Performing the analytization [22], we find

$$\mathcal{L}_2^{(2)}(Q^2) = \frac{4\pi}{b_0} \left[\mathcal{A}_1^{(2)}(Q^2) + c_1 \frac{4\pi}{b_0} f_{\mathcal{L}}(Q^2) \right], \quad (22)$$

where

$$f_{\mathcal{L}}(Q^2) = \sum_{n \geq 0} \left[\psi(2) \zeta(-n-1) - \frac{d\zeta(-n-1)}{dn} \right] \frac{[-\ln(Q^2/\Lambda^2)]^n}{\Gamma(n+1)} \quad (23)$$

and $\zeta(z)$ is the Riemann zeta-function. One can show (see for details [22]) that calculating $F_{\pi}^{\text{Fact}}(Q^2)$ under the proviso of the KS analytization provides an expression which is extremely stable against variations of the factorization scale. Indeed, varying the factorization scale from 1 GeV² to 10 GeV², the form factor changes by a mere 1.5 percent and reaches just the level of about 2.5 percent for a (hypothetical) factorization scale of 50 GeV². The sensitivity on the factorization scale using the Maximal Analytization is also a mild one, but the corresponding variation is, in round terms, two times larger. As regards the scale behavior of the form factor, both analytization schemes yield almost coincident results. Hence, $[Q^2 F_{\pi}^{\text{Fact}}(Q^2)]_{\text{KS}}^{An}$ in Fig. 2 cannot be differentiated from $[Q^2 F_{\pi}^{\text{Fact}}(Q^2)]_{\text{Max-An}}$. In concluding this analysis, using FAPT the dependence on all perturbative scheme and scale settings, including the factorization (evolution) scale, is diminished already at the NLO level.

We turn now to the second application, this time in Minkowski space: the decay of a scalar Higgs boson to a $b\bar{b}$ pair at the four-loop level of the quantity R_S from which one can obtain the width $\Gamma(H \rightarrow b\bar{b})$ [23]. In this case, we will encounter no ghost singularities—in contrast to the Euclidean space. However, the analytic continuation from the spacelike to the timelike region will entail so-called “kinematical” π^2 terms, whose contribution may become with increasing order of the perturbative expansion as important as the expansion coefficients.

To get a handle on the Higgs-boson decay, we consider the correlator of two scalar currents $J_b^S = \bar{\Psi}_b \Psi_b$ for bottom quarks with mass m_b , coupled to the scalar Higgs boson with mass M_H and where $Q^2 = -q^2$:

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iq \cdot x} \langle 0 | T[J_b^S(x) J_b^S(0)] | 0 \rangle. \quad (24)$$

Then, $R_S(s) = \text{Im } \Pi(-s - i\epsilon)/(2\pi s)$ and one can express the width in terms of R_S , i.e.,

$$\Gamma(H \rightarrow b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi} M_H m_b^2 (M_H) R_S(s = M_H^2). \quad (25)$$

One, finally, obtains R_S via the analytic continuation of the Adler function D into Euclidean space by applying on it the linear operation \mathbf{A}_M (equivalently, the integral transformation \hat{R}), according to the analytization machinery illustrated in Fig. 1 This means that one has to calculate the quantity [36]

$$\tilde{R}_S(s) \equiv \tilde{R}_S(Q^2 = s, \mu^2 = s) = 3m_b^2(s) \left[1 + \sum_{n \geq 1} r_n a_s^n(s) \right], \quad (26)$$

where the expansion coefficients r_n contain characteristic π^2 terms originating from the integral transformation \hat{R} of the powers of the logarithms appearing in \tilde{D}_S . The latter is related to $\tilde{R}_S(s, s)$ by means of a dispersion relation. Notice that these logarithms have two different sources: one is the running of a_s in \tilde{D}_S , while the other is related to the evolution of the heavy-quark mass $m_b^2(Q^2)$. As a result, the coefficients r_n in (26) are connected to the coefficients d_n in \tilde{D}_S (calculable in Euclidean space) and to a combination of the mass anomalous dimension γ_i and the β -function coefficients b_j , multiplied by π^2 powers [35–38].

The running mass in the l -loop approximation, $m_{(l)}$, can be cast in terms of the renormalization-group invariant quantity $\hat{m}_{(l)}$ to read

$$m_{(l)}^2(Q^2) = \hat{m}_{(l)}^2 [a_s(Q^2)]^{\nu_0} f_{(l)}(a_s(Q^2)), \quad (27)$$

where the expansion of $f_{(l)}(x)$ at the three-loop order is given by

$$\begin{aligned} f_{(l)}(a_s) = & 1 + a_s \frac{b_1}{2b_0} \left(\frac{\gamma_1}{b_1} - \frac{\gamma_0}{b_0} \right) + a_s^2 \frac{b_1^2}{16b_0^2} \left[\frac{\gamma_0}{b_0} - \frac{\gamma_1}{b_1} + 2 \left(\frac{\gamma_0}{b_0} - \frac{\gamma_1}{b_1} \right)^2 \right. \\ & \left. + \frac{b_0 b_2}{b_1^2} \left(\frac{\gamma_2}{b_2} - \frac{\gamma_0}{b_0} \right) \right] + O(a_s^3). \end{aligned} \quad (28)$$

We are now ready to consider the analytization of the Adler function

$$\tilde{D}_S(Q^2; \mu^2) = 3m_b^2(Q^2) \left[1 + \sum_{n \geq 1} d_n(Q^2/\mu^2) a_s^n(\mu^2) \right]. \quad (29)$$

Expanding the running mass in a power series, according to

$$m_{(l)}^2(Q^2) = \hat{m}_{(l)}^2 (a_s(Q^2))^{\nu_0} \left[1 + \sum_{m \geq 1} e_m^{(l)} (a_s(Q^2))^m \right], \quad (30)$$

and choosing $\mu^2 = Q^2$, we find

$$\begin{aligned} [3\hat{m}_b^2]_{(l)}^{-1} \tilde{D}_S^{(l)}(Q^2) &= \left(a_s^{(l)}(Q^2)\right)^{\nu_0} + \sum_{n \geq 1}^l d_n \left(a_s^{(l)}(Q^2)\right)^{n+\nu_0} \\ &+ \sum_{m \geq 1}^{\infty} \Delta_m^{(l)} \left(a_s^{(l)}(Q^2)\right)^{m+\nu_0} \end{aligned} \quad (31)$$

with

$$\Delta_m^{(l)} = e_m^{(l)} + \sum_{k \geq 1}^{\min[l, m-1]} d_k e_{m-k}^{(l)}. \quad (32)$$

Note that we have purportedly separated the mass-evolution effects (collected in the third term of Eq. (31)) from the original series expansion of D (truncated at $n = l$), the latter being represented by the second term on the RHS of Eq. (31). In practice, for $Q \geq 2$ GeV, i.e., for $\alpha_s \leq 0.4$, the truncation at $m = l + 4$ of the summation (30) produces a truncation error much smaller than 1 percent.

Finally, we obtain $\tilde{R}_S^{\text{MFAPT}}$ from the quantity $\tilde{D}_S^{(l)}(Q^2)$ by applying the analytization operation \mathbf{A}_M :

$$\begin{aligned} \tilde{R}_S^{(l)\text{MFAPT}} &= \mathbf{A}_M[D_S^{(l)}] \\ &= 3\hat{m}_{(l)}^2 \left[\mathbf{a}_{\nu_0}^{(l)} + \sum_{n \geq 1}^l d_n \mathbf{a}_{n+\nu_0}^{(l)} + \sum_{m \geq 1} \Delta_m^{(l)} \mathbf{a}_{m+\nu_0}^{(l)} \right], \end{aligned} \quad (33)$$

where we have used the short-hand notation

$$[a_s(s)^\nu]_{\text{an}} = \mathbf{a}_\nu^{(l)}(s) \equiv \left(\frac{4}{b_0}\right)^\nu \mathfrak{A}_\nu^{(l)}(s) \quad (34)$$

and $b_0 = \frac{11}{3}C_A - \frac{4}{3}T_R N_f$ with $C_A = N_c = 3$, $T_R = \frac{1}{2}$. The above expression contains, by means of the coefficients $\Delta_n^{(l)}$ ($e_k^{(l)}$) and the couplings $\mathbf{a}_{n+\nu_0}^{(l)}$, all renormalization-group terms contributing to this order, while the resummed π^2 terms are integral parts of the *analytic* couplings by construction.

The results for the quantity $\tilde{R}_S(M_H^2)$, calculated within different approaches in the $\overline{\text{MS}}$ scheme, versus the Higgs mass M_H are illustrated in Fig. 3. The long-dashed curve in this figure shows the predictions obtained by Baikov, Chetyrkin, and Kühn [35] employing standard perturbative QCD at the $l = 4$ loop level of expansion. The solid curve next to it represents

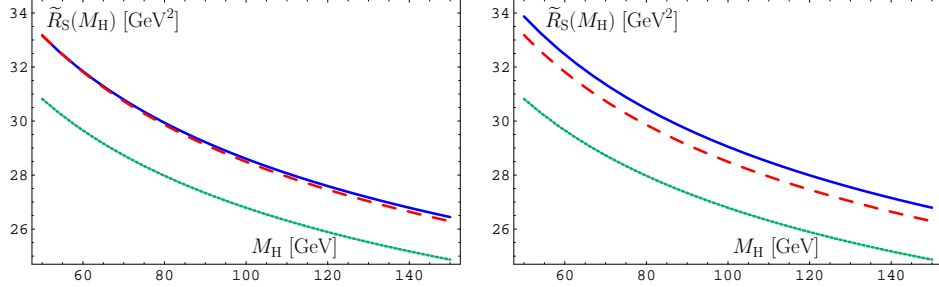


Fig. 3: Illustration of the calculation of the perturbative series of the quantity $\tilde{R}_S(M_H^2)$ in different approaches within the $\overline{\text{MS}}$ scheme: Standard perturbative QCD [35, 36] at the loop level $l = 4$ (dashed red line with $\Lambda_{N_f=5} = 231$ MeV), BKM estimates, by taking into account the $O((a_s)^{\nu_0} A_4(a_s))$ -terms, [37]—(dotted green line with $\Lambda_{N_f=5} = 111$ MeV), and MFAPT from Eq. (33) for $N_f = 5$ (solid blue line), displayed for two different loop orders: $l = 2$ (left panel, $\Lambda_{N_f=5} = 263$ MeV) and $l = 3$ (right panel with $\Lambda_{N_f=5} = 261$ MeV). The value of $\Lambda_{N_f=5}$ MeV in all cases corresponds to $\mathfrak{A}_1^{(1)}(s = m_Z^2; N_f = 5) = 0.120$.

the outcome of the FAPT machinery (cf. (33)), including in the second sum all evolution effects up to $m = l + 4$ and fixing the active flavor number to $N_f = 5$. Bear in mind that the π^2 terms, induced through the analytic continuation, are contained in the expansion coefficients $\mathfrak{a}_{m+\nu_0}^{(l)}$. On the other hand, the contributions of the higher-loop renormalization-group dependent terms are accumulated in the coefficients $\Delta_m^{(l)}$ by means of the parameters γ_i and b_j . It is obvious that for this observable the standard perturbative QCD approach and FAPT yield similar predictions, starting with the two-loop running. The reason for the slightly larger FAPT prediction lies in the fact that the coefficients \mathfrak{a}_ν contain the resummed contribution of an *infinite* series of π^2 -terms that renders them ultimately smaller than the corresponding powers of the standard coupling. [The interested reader is referred to [23] for further details.] Finally, the lower (green dotted) curve by about 8% in both panels of Fig. 3 gives the estimate of Broadhurst, Kataev, and Maxwell (BKM) [37], which relies upon the so-called “naive non-Abelianization” and an optimized power-series expansion that makes use of the “contour integration technique”.³ [Some more technical remarks can be found in [23], where the common elements between this approach and FAPT are worked out.]

³We thank A. L. Kataev for useful remarks pertaining to this figure in [23].

5 Conclusions

The generalized KS analyticity requirement [17] has proven successful in describing hadronic observables at the partonic level for a variety of reactions. Although this requirement has to be extra postulated, it is the one that provides a natural extension of the analyticity demand on the running coupling, proposed by Shirkov and Solovtsov [2], giving us a much broader understanding of analytization. We have shown that including into the dispersion relations the contributions stemming from all terms that affect the spectral density (even though these terms do not influence the nature of the ghost singularities of the standard power-series perturbative expansion), makes it possible to treat processes containing two large momentum scales. Such additional scales, like the factorization or the evolution scale, enter in the form of typical logarithms whose incorporation into the spectral density naturally amounts to non-integer (fractional) powers of the coupling. This analytization formalism—Fractional Analytic Perturbational Theory, developed in [21–23] on the theoretical basis of [17]—works equally well in both the space-like region (Euclidean space) as well as the timelike region (Minkowski space). In the first case, the obtained expressions for the hadronic observables are singularity-free and turn out to be insensitive to the renormalization scheme and scale adopted, while bearing little sensitivity to the factorization scale, as well. In the timelike regime—where ghost singularities are absent—a better stability is achieved in terms of expansion coefficients up to a high loop order that in situ resum all π^2 terms induced by analytic continuation.

In this short exposition we have not been exhaustive. We note in passing that we have derived closed-form expressions for the analytic-coupling images at the one-loop level in the spacelike [21] and in the timelike region, and further approximate expressions at the two-loop level [23], the latter supported by exact numerical results [11]. What is perhaps more, our approach provides a handle on the computation of power corrections and their coefficients to different hadronic reactions. We have already obtained leading-order power corrections to the pion’s electromagnetic form factor and to the cross section of the Drell-Yan process into a lepton pair [17, 18], which put the developed scheme into a larger theoretical and phenomenological context.

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